Chance-Constrained AC Optimal Power Flow: Modelling and Solution Approaches

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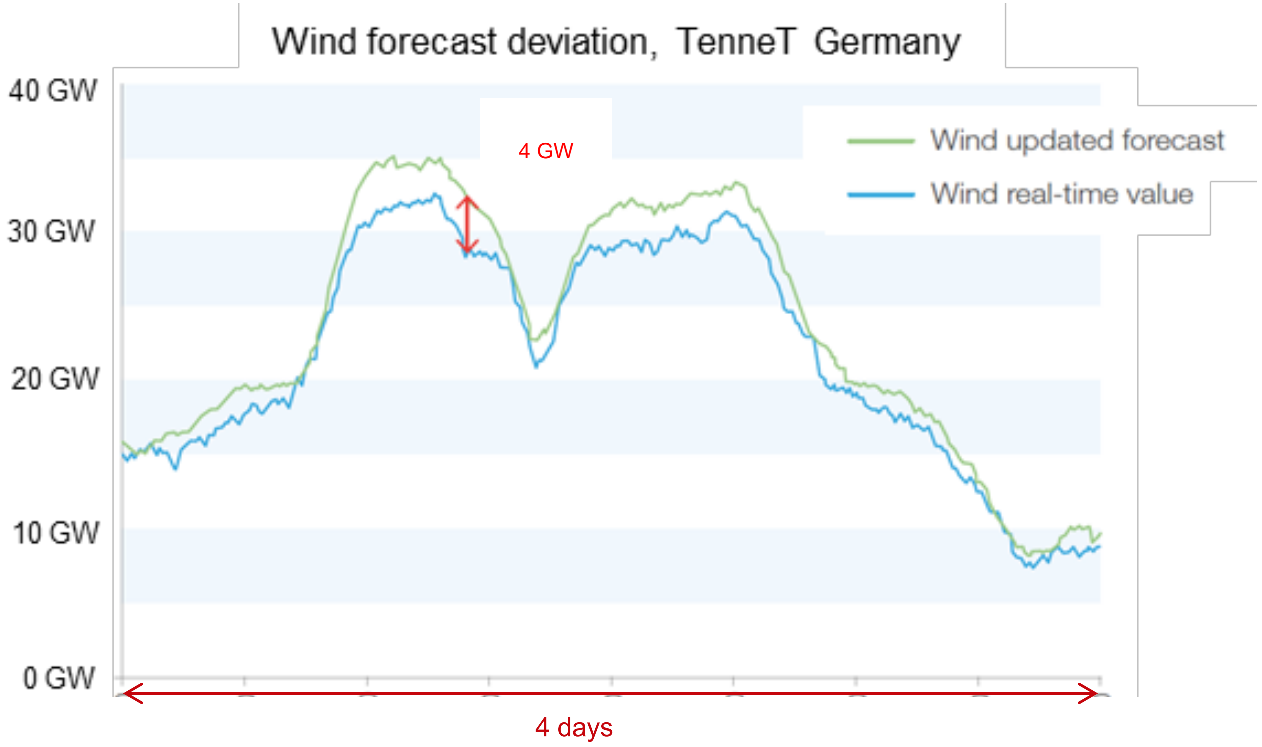
Power Line.

Power Line!

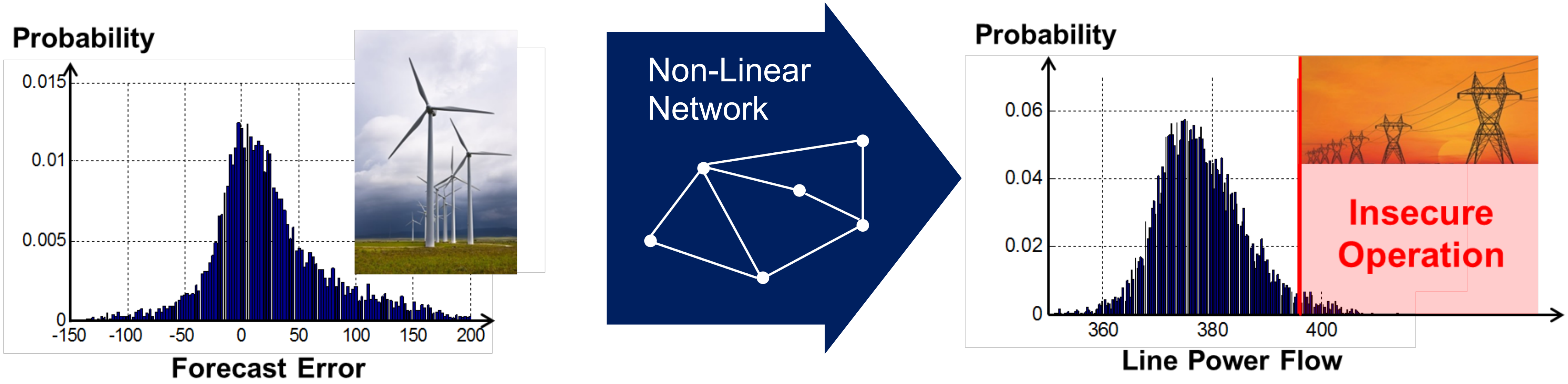
  

Joint work with Sidhant Misra (LANL), Tillmann Mühlpfordt (KIT) and Göran Andersson (ETH)

Wind power in Germany



Impact of uncertainty



What’s the problem?

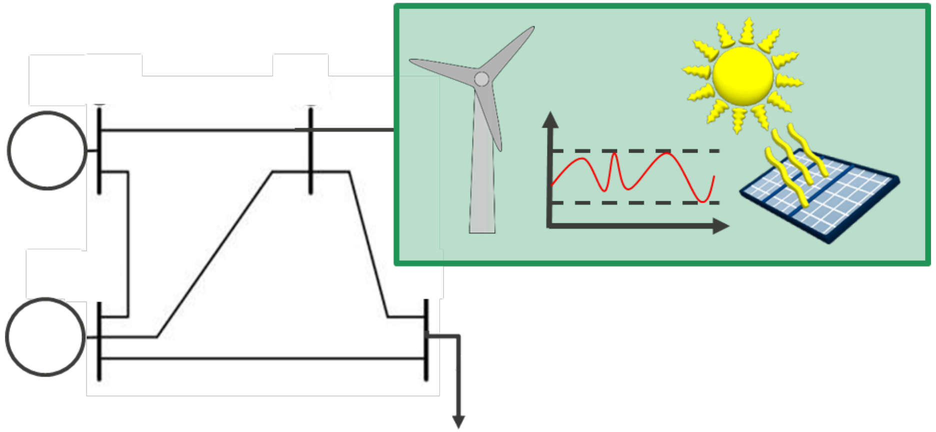
Chance-constrained AC Optimal Power Flow

What’s the problem?

security against
uncertain injections

$$\underbrace{\textbf{Chance-constrained}}$$

Chance-constrained AC Optimal Power Flow



What's the problem?

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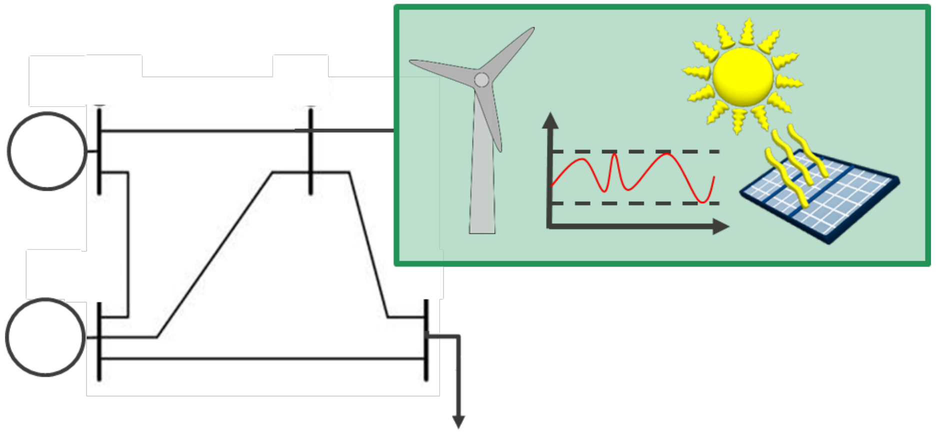
Chance-constrained AC Optimal Power Flow



accurate system model

non-linear equations

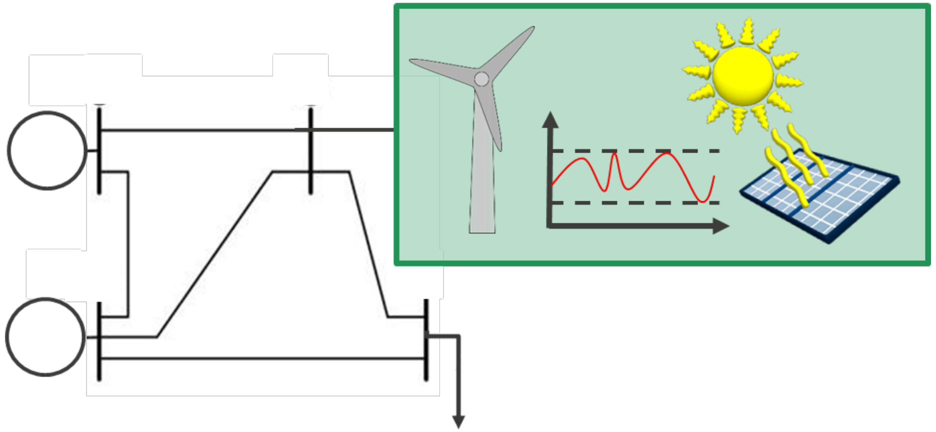
→ **non-convex** constraints

What’s the problem?

security against
uncertain injections

optimality =
economic efficiency

Chance-constrained AC Optimal Power Flow



accurate system model

non-linear equations

→ **non-convex** constraints

What's the problem?

security against uncertain injections

optimality = economic efficiency

$$\underbrace{\text{Chance-constrained}}_{\text{security against uncertain injections}}\ \text{AC}\ \text{Optimal}\ \text{Power Flow}$$

Chance-constrained AC Optimal Power Flow

Methods to guarantee both chance-constraint feasibility and optimality subject to non-linear AC constraints?

What's the problem?

security against uncertain injections

optimality = economic efficiency

Robust and Stochastic AC Optimal Power Flow

scalable!

Methods to guarantee both chance-constraint feasibility and optimality subject to non-linear AC constraints?

A brief overview of literature on AC OPF with uncertainty

(There is not a lot…)

- Worst-case scenario for non-convex AC OPF — [Capitanescu, Fliscounakis, Panciatici, & Wehenkel '12]
  - No guarantees due to non-convexity
- Linearization of AC power flow equations — [Dall'Anese, Baker & Summers '16], [Roald & Andersson '17], [Lubin, Dvorkin, Roald, '19] …
  - Accurate only close to linearization point
- Chance-constrained polynomial chaos expansion — [Mühlpfort, Roald, Hagenmeyer, Faulwasser & Misra, preprint]
  - Scalability and good reformulations
- SDP-based chance-constraint reformulations — [Weisser, Roald & Misra, preprint]
  - Scalability !!!
- Convex relaxation + linearization of voltage products — [Vrakopoulou at al, '13], [Venzke et al '17]
  - Are not exact
- Convex inner approximations — [Louca & Bitar '17], [Misra et al, 2017]
  - Does not handle equality constraints = requires controllable injections at every bus
- Convex relaxation + two/multi-stage robust program — [Nasri, Kazempour, Conejo, & Ghandhari '16] [Phan & Ghosh '14], [Lorca & Sun '17]
  - Lower bound (no guarantees)
- Robust bounds on uncertainty impact — [Molzahn and Roald '18], [Molzahn and Roald '19]
  - Upper bounds (?)

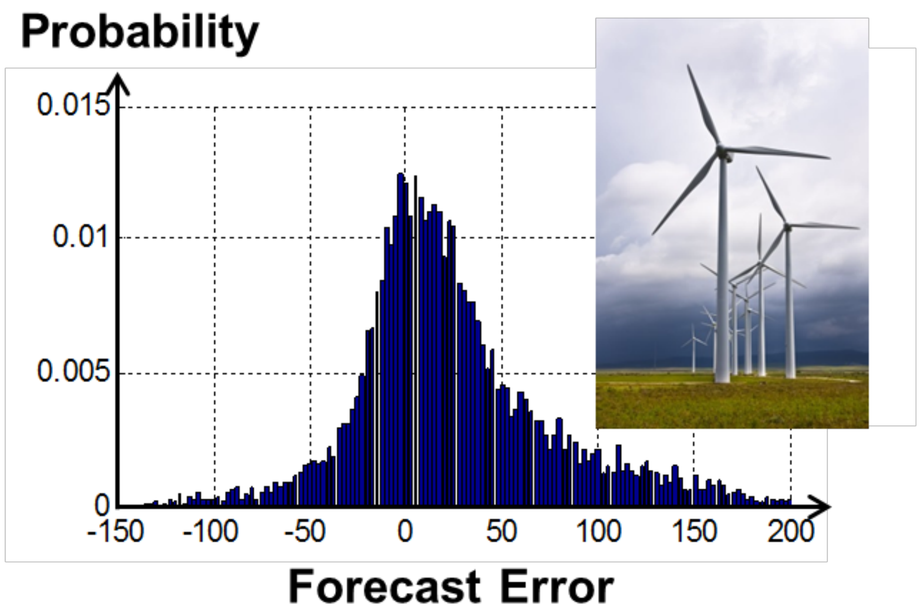
Outline

- A complicated model
- A simple chance constraint
- Solution approaches

Renewable energy uncertainty

- Changes in power generation $\boldsymbol{p_{inj}}$ due to renewable forecast errors $\boldsymbol{\omega}$:

$$\boldsymbol{p_{inj}}(\boldsymbol{\omega}) = \boldsymbol{\hat{p}_{inj}} + \boldsymbol{\omega}$$



- Assumptions on ω :
  - **Known** and finite mean and covariance $\mu_\omega, \Sigma_\omega$

- Reactive power changes: $\boldsymbol{q_{inj}}(\boldsymbol{\omega}) = \boldsymbol{\hat{q}_{inj}} + \boldsymbol{\gamma\omega}$

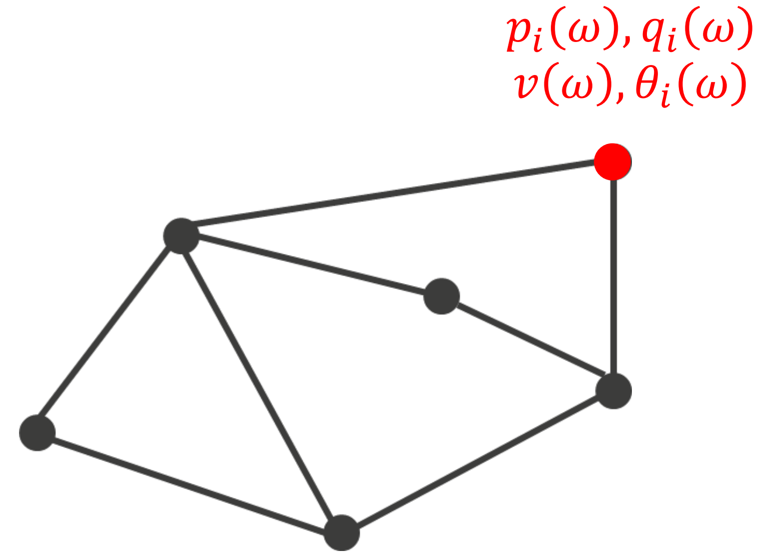
Network model

- AC power flow equations: Conservation of power at each node

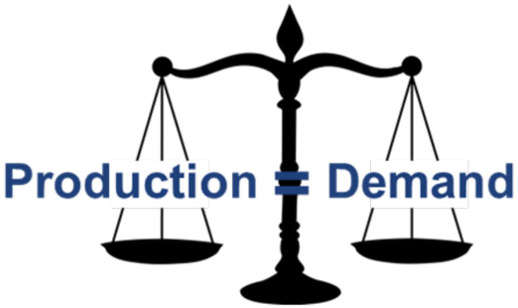
$$p_{inj,i}(\omega) = v_i(\omega) \sum_{k=1}^{n} v_k(\omega) \Big[\mathbf{G}_{ik} \cos\Big(\theta_i(\omega) - \theta_k(\omega)\Big) + \mathbf{B}_{ik} \sin\Big(\theta_i(\omega) - \theta_k(\omega)\Big) \Big]$$

$$q_{inj,i}(\omega) = v_i(\omega) \sum_{k=1}^{n} v_k(\omega) \Big[\mathbf{G}_{ik} \sin\Big(\theta_i(\omega) - \theta_k(\omega)\Big) - \mathbf{B}_{ik} \cos\Big(\theta_i(\omega) - \theta_k(\omega)\Big) \Big]$$

$$\theta_{ref}(\omega) = 0$$



Recourse actions



(we would like to optimize α)

- Affine recourse policy for active power balancing

$$p_{G,i}(\omega) = p_{G0,i} - \alpha_i \left(\sum_{k \in \mathcal{N}} \omega_k - \delta p\,(\omega) \right)$$

- Constant voltage magnitudes at generators

$$v_G(\omega) = v_{G0}$$

AC Optimal Power Flow Formulation

$$\min \sum_{i\in\mathcal{G}} \left(c_{2,i}\,(p_{G0,i})^2 + c_{1,i}\,p_{G0,i} + c_{0,i}\right)$$

Cost for expected operating point

$$\text{subject to} \quad \forall i \in \mathcal{N},\ \forall(\ell, m) \in \mathcal{L}$$

$$p_{G,i}(\omega) + \hat{p}_{inj,i}(\omega) + \omega_i$$
$$= v_i(\omega)\sum_{k=1}^{n} v_k(\omega)\left[\mathbf{G}_{ik}\cos\left(\theta_i(\omega) - \theta_k(\omega)\right) + \mathbf{B}_{ik}\sin\left(\theta_i(\omega) - \theta_k(\omega)\right)\right]$$
$$q_{G,i}(\omega) + \hat{q}_{inj,i}(\omega) + \gamma\,\omega_i$$
$$= v_i(\omega)\sum_{k=1}^{n} v_k(\omega)\left[\mathbf{G}_{ik}\sin\left(\theta_i(\omega) - \theta_k(\omega)\right) - \mathbf{B}_{ik}\cos\left(\theta_i(\omega) - \theta_k(\omega)\right)\right]$$

AC power flow equations

$$p_{G,k}(\omega) = p_{G0,k} - \alpha\left(\sum_{i\in\mathcal{N}}\omega_i - \delta p\,(\omega)\right),$$
$$v_k(\omega) = v_{G0,k}, \qquad \forall k \in \mathcal{G}$$

Generation and voltage control policies

$$p_{G,i}^{min} \leq p_{G,i}(\omega) \leq p_{G,i}^{max}$$
$$q_{G,i}^{min} \leq q_{G,i}(\omega) \leq q_{G,i}^{max}$$
$$v_i^{min} \leq v_i(\omega) \leq v_i^{max}$$
$$|i_{\ell m}(\omega)| \leq i_{\ell m}^{max}, \quad |i_{m\ell}(\omega)| \leq i_{\ell m}^{max}$$

Generation, voltage and transmission limits

Chance-constrained AC Optimal Power Flow

$$\min \sum_{i\in\mathcal{G}} \left(c_{2,i}\,(p_{G0,i})^2 + c_{1,i}\,p_{G0,i} + c_{0,i}\right)$$

Cost for expected operating point

$$\text{subject to} \quad (\forall i \in \mathcal{N},\ \forall(\ell,m)\in\mathcal{L},\ \forall\omega\in\mathcal{W})$$

$$p_{G,i}(\omega) + \hat{p}_{inj,i}(\omega) + \omega_i = v_i(\omega)\sum_{k=1}^{n} v_k(\omega)\left[\mathbf{G}_{ik}\cos\left(\theta_i(\omega)-\theta_k(\omega)\right) + \mathbf{B}_{ik}\sin\left(\theta_i(\omega)-\theta_k(\omega)\right)\right]$$

$$q_{G,i}(\omega) + \hat{q}_{inj,i}(\omega) + \gamma\,\omega_i = v_i(\omega)\sum_{k=1}^{n} v_k(\omega)\left[\mathbf{G}_{ik}\sin\left(\theta_i(\omega)-\theta_k(\omega)\right) - \mathbf{B}_{ik}\cos\left(\theta_i(\omega)-\theta_k(\omega)\right)\right]$$

Robust AC power flow equations

$$p_{G,k}(\omega) = p_{G0,k} - \alpha\left(\sum_{i\in\mathcal{N}} \omega_i - \delta p\,(\omega)\right),$$

$$v_k(\omega) = v_{G0,k}, \qquad \forall k \in \mathcal{G}$$

Generation and voltage control policies

$$\mathbb{P}(\ p_{G,i}^{min} \le p_{G,i}(\omega) \le p_{G,i}^{max}\ ) \ge 1-\boldsymbol{\varepsilon}$$

$$\mathbb{P}(\ q_{G,i}^{min} \le q_{G,i}(\omega) \le q_{G,i}^{max}\ ) \ge 1-\boldsymbol{\varepsilon}$$

$$\mathbb{P}(\ v_i^{min} \le v_i(\omega) \le v_i^{max}\ ) \ge 1-\boldsymbol{\varepsilon}$$

$$\mathbb{P}(\ |i_{\ell m}(\omega)| \le i_{\ell m}^{max},\ \ |i_{m\ell}(\omega)| \le i_{\ell m}^{max}\ ) \ge 1-\boldsymbol{\varepsilon}$$

Single chance constraints for generation, voltage and transmission limits

Chance-constrained AC Optimal Power Flow

Why robust power flow equations?

$$\left.\begin{aligned} & p_{G,i}(\omega) + \hat{p}_{inj,i}(\omega) + \omega_i \\ & \quad = v_i(\omega) \sum_{k=1}^{n} v_k(\omega) \left[\mathbf{G}_{ik} \cos\Big(\theta_i(\omega) - \theta_k(\omega)\Big) + \mathbf{B}_{ik} \sin\Big(\theta_i(\omega) - \theta_k(\omega)\Big) \right] \\ & q_{G,i}(\omega) + \hat{q}_{inj,i}(\omega) + \gamma\, \omega_i \\ & \quad = v_i(\omega) \sum_{k=1}^{n} v_k(\omega) \left[\mathbf{G}_{ik} \sin\Big(\theta_i(\omega) - \theta_k(\omega)\Big) - \mathbf{B}_{ik} \cos\Big(\theta_i(\omega) - \theta_k(\omega)\Big) \right] \end{aligned}\right\}$$

Robust AC power flow equations

If the power flow equations are not satisfied, the model does not make sense.

Chance-constrained AC Optimal Power Flow

How robust power flow equations?

$$\left.\begin{aligned}
&p_{G,i}(\omega) + \hat{p}_{inj,i}(\omega) + \omega_i \\
&\quad = v_i(\omega) \sum_{k=1}^{n} v_k(\omega) \left[\mathbf{G}_{ik} \cos\Big(\theta_i(\omega) - \theta_k(\omega)\Big) + \mathbf{B}_{ik} \sin\Big(\theta_i(\omega) - \theta_k(\omega)\Big) \right] \\
&q_{G,i}(\omega) + \hat{q}_{inj,i}(\omega) + \gamma\,\omega_i \\
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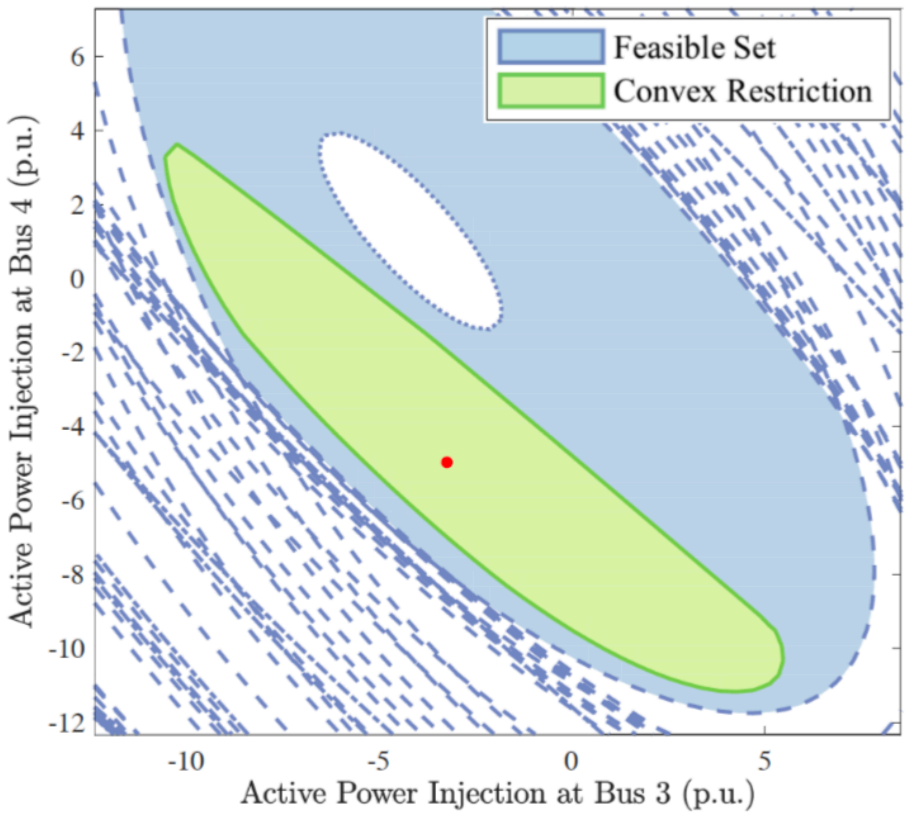
\end{aligned}\right\} \text{Robust AC power flow equations}$$

Chance-constrained AC Optimal Power Flow

How robust power flow equations?

Convex restriction =
convex *inner* approximation

Convex quadratic constraints



D Lee, HD Nguyen, K Dvijotham, K Turitsyn, “Convex restriction of AC power flow feasibility set”, arXiv preprint arXiv:1803.00818

D Lee, K Turitsyn, D K Molzahn, L Roald, “Feasible Path Identification in Optimal Power Flow with Sequential Convex Restriction”, https://arxiv.org/abs/1906.09483

Chance-constrained AC Optimal Power Flow

$$\min \sum_{i \in \mathcal{G}} \left(c_{2,i} \, (p_{G0,i})^2 + c_{1,i} \, p_{G0,i} + c_{0,i} \right)$$

Cost for expected operating point

$$\text{subject to} \quad (\forall i \in \mathcal{N}, \ \forall (\ell, m) \in \mathcal{L}, \ \forall \omega \in \mathcal{W})$$

$$p_{G,i}(\omega) + \hat{p}_{inj,i}(\omega) + \omega_i = v_i(\omega) \sum_{k=1}^{n} v_k(\omega) \left[\mathbf{G}_{ik} \cos\left(\theta_i(\omega) - \theta_k(\omega)\right) + \mathbf{B}_{ik} \sin\left(\theta_i(\omega) - \theta_k(\omega)\right) \right]$$

$$q_{G,i}(\omega) + \hat{q}_{inj,i}(\omega) + \gamma \, \omega_i = v_i(\omega) \sum_{k=1}^{n} v_k(\omega) \left[\mathbf{G}_{ik} \sin\left(\theta_i(\omega) - \theta_k(\omega)\right) - \mathbf{B}_{ik} \cos\left(\theta_i(\omega) - \theta_k(\omega)\right) \right]$$

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Generation and voltage control policies

$$\mathbb{P}(\, p_{G,i}^{min} \leq p_{G,i}(\omega) \leq p_{G,i}^{max} \,) \geq 1 - \varepsilon$$

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$$\mathbb{P}(\, |i_{\ell m}(\omega)| \leq i_{\ell m}^{max}, \ |i_{m\ell}(\omega)| \leq i_{\ell m}^{max} \,) \geq 1 - \varepsilon$$

Single chance constraints for generation, voltage and transmission limits

Chance-constrained AC Optimal Power Flow

Why single chance constraints?

Modelling perspective:

Joint – probability of having a peaceful afternoon at work

Single – easier to assign risk to certain components

Solution perspective:

Joint – computational tractability, conservativeness

Single – easier, less safe

$$\left.\begin{aligned}
&\mathbb{P}(\, p_{G,i}^{min} \leq p_{G,i}(\omega) \leq p_{G,i}^{max} \,) \geq 1-\varepsilon \\
&\mathbb{P}(\, q_{G,i}^{min} \leq q_{G,i}(\omega) \leq q_{G,i}^{max} \,) \geq 1-\varepsilon \\
&\mathbb{P}(\, v_{i}^{min} \leq v_{i}(\omega) \leq v_{i}^{max} \,) \geq 1-\varepsilon \\
&\mathbb{P}(\, |i_{\ell m}(\omega)| \leq i_{\ell m}^{max},\ |i_{m\ell}(\omega)| \leq i_{\ell m}^{max} \,) \geq 1-\varepsilon
\end{aligned}\right\}$$

Single chance constraints for generation, voltage and transmission limits

Chance-constrained AC Optimal Power Flow

Why single chance constraints?

Many constraints

~ 16 million for a realistic system

(Polish test case with security constraints)

High dimensional ω

~ 941 uncertain loads (Polish test case)

Possible to control joint violation probability using single constraints

	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.1$
Max. ϵ_{emp}	0.013	0.044	0.092
Joint ϵ_J	0.065	0.137	0.219

$$\left.\begin{aligned}
&\mathbb{P}(\, p_{G,i}^{min} \leq p_{G,i}(\omega) \leq p_{G,i}^{max}\,) \geq 1 - \varepsilon \\
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\end{aligned}\right\}$$

Single chance constraints for generation, voltage and transmission limits

Outline

- A complicated model
- A simple chance constraint
- Solution approaches

Moment-based Reformulation

$$\mathbb{P}(i(x,\omega) \le i^{max}) \ge 1 - \epsilon$$

$$\boldsymbol{\mu_i(x,\omega)} + \rho(\epsilon)\boldsymbol{\sigma_i(x,\omega)} \le i^{max}$$

Exact reformulation if $\omega \sim \mathcal{N}(\mu_\omega, \Sigma_\omega)$
and $\rho(\epsilon) = \Phi^{-1}(1-\epsilon)$

Moment-based Reformulation

Bad news!

$$\mathbb{P}(i(x,\omega) \leq i^{max}) \geq 1 - \epsilon$$

$$\boldsymbol{\mu_i(x,\omega)} + \rho(\epsilon)\boldsymbol{\sigma_i(x,\omega)} \leq i^{max}$$

Exact reformulation if $\omega \sim \mathcal{N}(\mu_\omega, \Sigma_\omega)$ and $\rho(\epsilon) = \Phi^{-1}(1-\epsilon)$

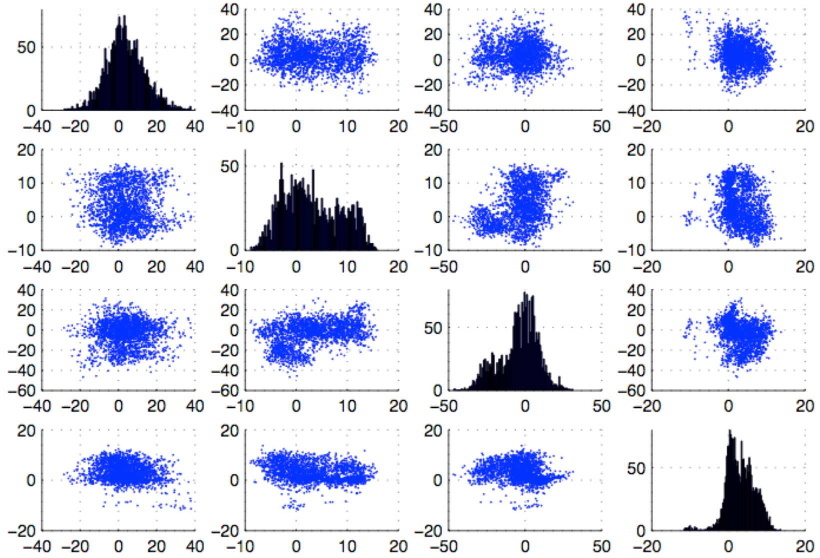


Fig. 2. Forecast errors for 4 selected nodes of case study. The diagonal plots show the histograms of the forecast errors (x-axis: deviation in MW, y-axis: number of occurences), while the off-diagonal plots show the scatter plots between two corresponding forecast errors (x- and y-axis: deviation in MW).

Data is NOT normally distributed…

[Roald, Oldewurtel, Van Parys & Andersson, arxiv '15]

Moment-based Reformulation

$$\mathbb{P}(i(x,\omega) \leq i^{max}) \geq 1 - \epsilon$$

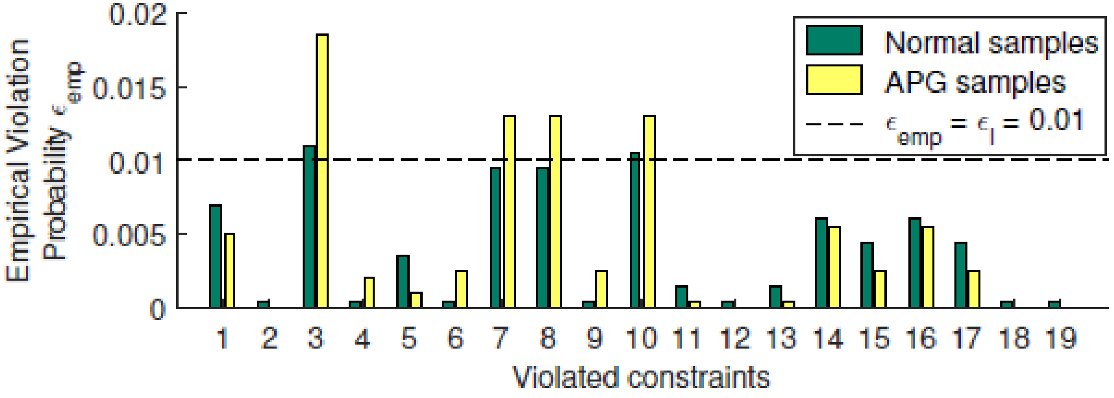
$$\boldsymbol{\mu_i(x,\omega)} + \rho(\epsilon)\boldsymbol{\sigma_i(x,\omega)} \leq i^{max}$$

Exact reformulation if $\omega \sim \mathcal{N}(\mu_\omega, \Sigma_\omega)$ and $\rho(\epsilon) = \Phi^{-1}(1-\epsilon)$

Good news!

In practice, normal distributions seem to provide very reasonable approximations

Concentration (?)



[Roald, Misra, Krause Andersson, 2017]

Moment-based Reformulation

Good news!

$$\mathbb{P}(i(x,\omega) \le i^{max}) \ge 1 - \epsilon$$

We can derive (conservative) values for $\rho(\epsilon)$ for (families of) non-normal distributions which share the mean and covariance $\mu_\omega, \Sigma_\omega$

Unimodality, …

$$\boldsymbol{\mu_i(x,\omega)} + \rho(\epsilon)\boldsymbol{\sigma_i(x,\omega)} \le i^{max}$$

Exact reformulation if $\omega \sim \mathcal{N}(\mu_\omega, \Sigma_\omega)$
and $\rho(\epsilon) = \Phi^{-1}(1-\epsilon)$

Interpretability

$$\boldsymbol{\mu_i}(\boldsymbol{x}, \boldsymbol{\omega}) + \rho(\epsilon)\boldsymbol{\sigma_i}(\boldsymbol{x}, \boldsymbol{\omega}) \leq i^{max}$$

$$\underbrace{\boldsymbol{\mu_i}(\boldsymbol{x}, \boldsymbol{\omega}) \leq i^{max}}_{\text{deterministic constraint}} - \underbrace{\rho(\epsilon)\boldsymbol{\sigma_i}(\boldsymbol{x}, \boldsymbol{\omega})}_{\text{“uncertainty margin”}}$$

How do I find $\boldsymbol{\mu_i}(\boldsymbol{x}, \boldsymbol{\omega})$ and $\boldsymbol{\sigma_i}(\boldsymbol{x}, \boldsymbol{\omega})$?

1. Linearize the AC power flow

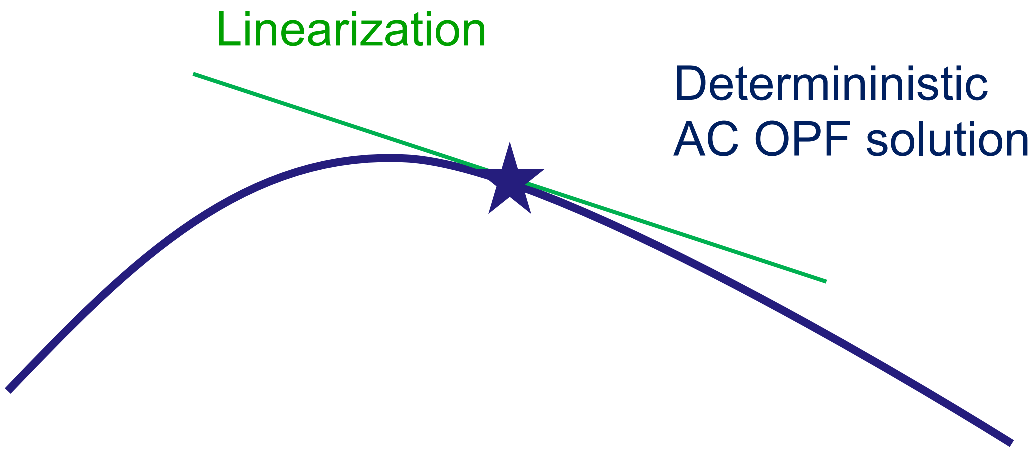
[Dall'Anese, Baker & Summers '16], [Lubin, Dvorkin & Roald '18], …

Taylor expansion for x and ω

$$v(x,\omega) \approx v(x_0,0) + \frac{dv}{dx}\Big|_{(x_o,0)}(x - x_0) + \frac{dv}{d\omega}\Big|_{(x_o,0)}\omega$$

$$\mu_v(x,\omega) \approx v(x_0,0) + \frac{dv}{dx}\Big|_{(x_o,0)}(x - x_0)$$

$$\sigma_v(x,\omega) \approx \sqrt{\left(\frac{dv}{d\omega}\Big|_{(x_o,0)}\right)\Sigma_\omega\left(\frac{dv}{d\omega}\Big|_{(x_o,0)}^T\right)}$$



2. Partially linearize the AC power flow

[Schmidli, Roald, Chatzivasileiadis and Andersson ‘16]
[Roald and Andersson ‘18]

Taylor expansion for ~~x and~~ ω

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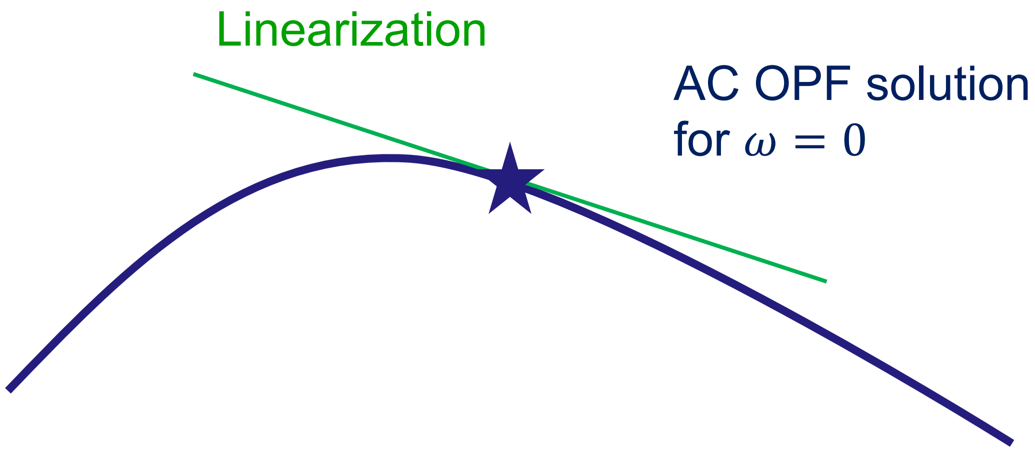

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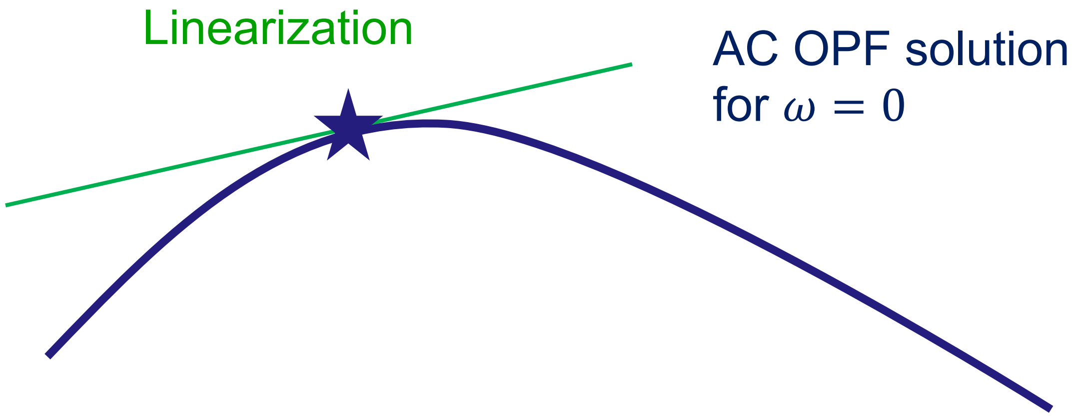

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$$v(x,\omega) \approx v(x,0) + \frac{dv}{d\omega}|_{(x_o,0)}\omega$$

$$\mu_v(x,\omega) \approx v(x,0)$$

$$\sigma_v(x,\omega) \approx \sqrt{\left(\frac{dv}{d\omega}|_{(x_o,0)}\right)\Sigma_\omega\left(\frac{dv}{d\omega}|^T_{(x_o,0)}\right)}$$

3. Polynomial Chaos Expansion

[Mühlpfort, Roald, Hagenmeyer, Faulwasser and Misra, accepted, '19]

1. Build a polynomial basis based on orthogonal polynomials from random variables

2. Express power flow and decision variables in terms of basis polynomials with unknown coefficients

3. Truncate at finite dimension

4. Solve optimal power flow with polynomials as constraints

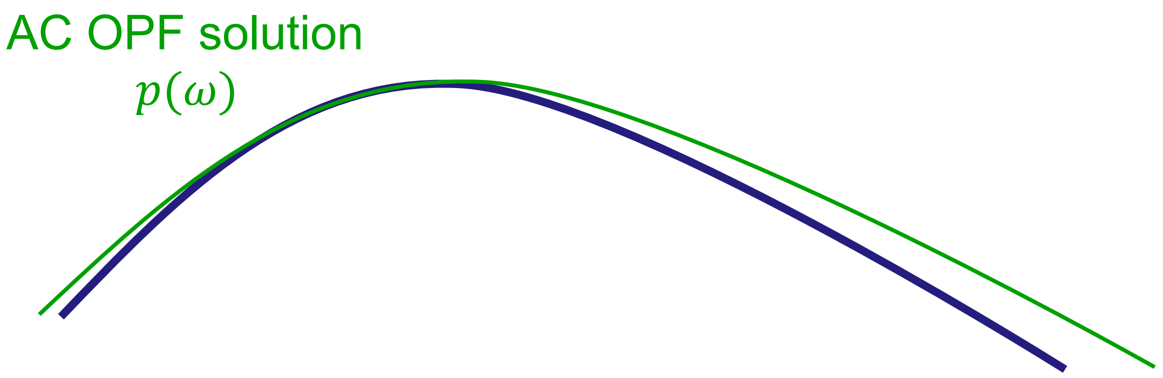

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Table I. REFORMULATIONS OF POWER FLOW EQUATIONS AND MOMENTS IN TERMS OF PCE COEFFICIENTS.

Rectangular power flow in terms of PCE coefficients with $i \in \mathcal{N}$, $k \in \mathcal{K}$
$\langle \psi_k, \psi_k \rangle (p^{\mathrm{g}}_{i,k} - p^{\mathrm{u}}_{i,k}) = \sum_{j \in \mathcal{N}} \sum_{k_1, k_2 \in \mathcal{K}} \langle \psi_{k_1} \psi_{k_2}, \psi_k \rangle (G_{ij}(v^{\mathrm{re}}_{i,k_1} v^{\mathrm{re}}_{j,k_2} + v^{\mathrm{im}}_{i,k_1} v^{\mathrm{im}}_{j,k_2}) + B_{ij}(v^{\mathrm{im}}_{i,k_1} v^{\mathrm{re}}_{j,k_2} - v^{\mathrm{re}}_{i,k_1} v^{\mathrm{im}}_{j,k_2}))$
$\langle \psi_k, \psi_k \rangle (q^{\mathrm{g}}_{i,k} - q^{\mathrm{u}}_{i,k}) = \sum_{j \in \mathcal{N}} \sum_{k_1, k_2 \in \mathcal{K}} \langle \psi_{k_1} \psi_{k_2}, \psi_k \rangle (G_{ij}(v^{\mathrm{im}}_{i,k_1} v^{\mathrm{re}}_{j,k_2} - v^{\mathrm{re}}_{i,k_1} v^{\mathrm{im}}_{j,k_2}) - B_{ij}(v^{\mathrm{re}}_{i,k_1} v^{\mathrm{re}}_{j,k_2} + v^{\mathrm{im}}_{i,k_1} v^{\mathrm{im}}_{j,k_2}))$
Moments of squared line current magnitudes with $ij \in \mathcal{L}$, $v^{\mathrm{re}}_{ij,k} = v^{\mathrm{re}}_{i,k} - v^{\mathrm{re}}_{j,k}$, $v^{\mathrm{im}}_{ij,k} = v^{\mathrm{im}}_{i,k} - v^{\mathrm{im}}_{j,k}$
$\mathbb{E}[\mathsf{i}^2_{i\text{-}j}] = \lvert y^{\mathrm{br}}_{ij} \rvert^2 \sum_{k \in \mathcal{K}} \langle \psi_k, \psi_k \rangle ((v^{\mathrm{re}}_{ij,k})^2 + (v^{\mathrm{im}}_{ij,k})^2)$
$\sigma[\mathsf{i}^2_{i\text{-}j}]^2 = \lvert y^{\mathrm{br}}_{ij} \rvert^4 \sum_{k_1,k_2,k_3,k_4 \in \mathcal{K}} \langle \psi_{k_1} \psi_{k_2} \psi_{k_3}, \psi_{k_4} \rangle (v^{\mathrm{re}}_{i,k_1} v^{\mathrm{re}}_{ij,k_2} v^{\mathrm{re}}_{i,k_3} v^{\mathrm{re}}_{ij,k_4} + 2 v^{\mathrm{re}}_{ij,k_1} v^{\mathrm{re}}_{ij,k_2} v^{\mathrm{im}}_{ij,k_3} v^{\mathrm{im}}_{ij,k_4} + v^{\mathrm{im}}_{ij,k_1} v^{\mathrm{im}}_{ij,k_2} v^{\mathrm{im}}_{ij,k_3} v^{\mathrm{im}}_{ij,k_4}) - \mathbb{E}[\mathsf{i}^2_{i\text{-}j}]^2$
Moments of squared voltage magnitudes with $i \in \mathcal{N}$
$\mathbb{E}[\mathsf{v}^2_i] = \sum_{k \in \mathcal{K}} \langle \psi_k, \psi_k \rangle ((v^{\mathrm{re}}_{i,k})^2 + (v^{\mathrm{im}}_{i,k})^2)$
$\sigma[\mathsf{v}^2_i]^2 = \sum_{k_1,k_2,k_3,k_4 \in \mathcal{K}} \langle \psi_{k_1} \psi_{k_2} \psi_{k_3}, \psi_{k_4} \rangle (v^{\mathrm{re}}_{i,k_1} v^{\mathrm{re}}_{i,k_2} v^{\mathrm{re}}_{i,k_3} v^{\mathrm{re}}_{i,k_4} + 2 v^{\mathrm{re}}_{i,k_1} v^{\mathrm{re}}_{i,k_2} v^{\mathrm{im}}_{i,k_3} v^{\mathrm{im}}_{i,k_4} + v^{\mathrm{im}}_{i,k_1} v^{\mathrm{im}}_{i,k_2} v^{\mathrm{im}}_{i,k_3} v^{\mathrm{im}}_{i,k_4}) - \mathbb{E}[\mathsf{v}^2_i]^2$

Similar structure as power flow equations…

JUST MANY MORE!

When can we truncate?

3. Polynomial Chaos Expansion

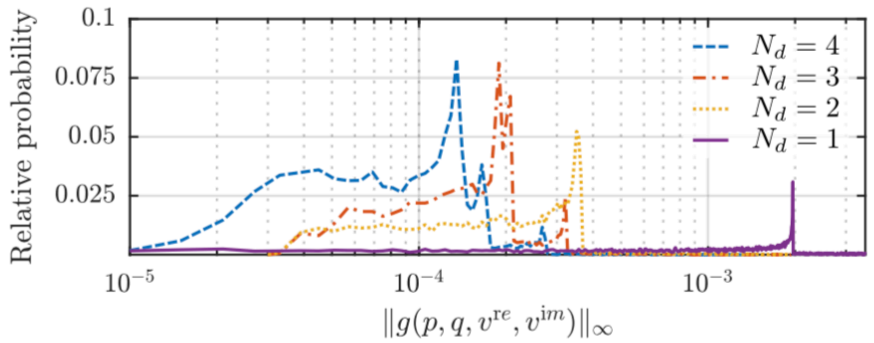
[Mühlpfort, Roald, Hagenmeyer, Faulwasser and Misra, accepted ‘19]

1. Build a polynomial basis based on orthogonal polynomials from random variables

2. Express power flow and decision variables in terms of basis polynomials with unknown coefficients

3. Truncate at finite dimension

4. Solve optimal power flow with polynomials as constraints



PCE bases of degree 2

(quadratic polynomials) already provide good results.

Comparison

1. Linearize the AC power flow

   ++ Computational speed

   - - Inaccuracy

2. Partially linearize the AC power flow

   + Easy to compute moments,

   + Computational speed

   - (less) inaccuracy

3. Polynomial Chaos Expansion

   + Efficient computation of moments

   + Accuracy

   - - Computational tractability
   (limited to small systems/
   few uncertainty sources)

Comparison

1. Linearize the AC power flow
   ++ Computational speed
   - - Inaccuracy

2. Partially linearize the AC power flow
   \+ Easy to compute moments,
   \+ Computational speed
   \- (less) inaccuracy

3. Polynomial Chaos Expansion
   \+ Efficient computation of moments
   \+ Accuracy
   - - Computational tractability
   (limited to small systems/
   few uncertainty sources)

Provide good approximations.

Linearization error ≈ Distribution error

In-sample testing (normal distribution)

	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.1$
Max. ϵ_{emp}	0.013	0.044	0.092
Joint ϵ_J	0.065	0.137	0.219

Out-of-sample testing (non-normal)

	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.1$
Max. ϵ_{emp}	0.014	0.054	0.093
Joint ϵ_J	0.074	0.145	0.233

Comparison

1. Linearize the AC power flow
   - ++ Computational speed
   - - - Inaccuracy

2. Partially linearize the AC power flow
   - \+ Easy to compute moments,
   - \+ Computational speed
   - \- (less) inaccuracy

Provide good approximations.

3. Polynomial Chaos Expansion
   - \+ Efficient computation of moments
   - \+ Accuracy
   - - - Computational tractability (limited to small systems/ few uncertainty sources)

How much better is

Polynomial Chaos Expansion?

Errors in Polynomial Chaos and Linearized AC

Linearized AC generally at least one order of magnitude larger errors.

Linearized AC introduces errors in estimating the mean!

Table IV. Error in the computed moments for the PCE method (PCE) and the linearization method (LIN) for 30-bus system.

AC vs.	s	p^g $\|\Delta\mu\|_\infty$	p^g $\|\Delta\sigma\|_\infty$	q^g $\|\Delta\mu\|_\infty$	q^g $\|\Delta\sigma\|_\infty$	v $\|\Delta\mu\|_\infty$	v $\|\Delta\sigma\|_\infty$	i_{i-j} $\|\Delta\mu\|_\infty$	i_{i-j} $\|\Delta\sigma\|_\infty$
PCE	0.05	1.8 E-5	0.6 E-5	1.7 E-5	0.7 E-5	0.3 E-5	0.4 E-5	5.1 E-5	3.9 E-5
	0.10	10.1 E-5	0.4 E-5	2.0 E-5	2.0 E-5	2.2 E-5	1.0 E-5	33.4 E-5	5.7 E-5
	0.15	2.9 E-5	19.8 E-5	10.7 E-5	6.4 E-5	3.8 E-5	1.1 E-5	19.3 E-5	12.1 E-5
lin. AC	0.05	431.2 E-5	4.0 E-5	0.131	81.1 E-5	108.4 E-5	2.3 E-5	4901.5 E-5	104.0 E-5
	0.10	411.0 E-5	12.9 E-5	0.136	294.3 E-5	105.8 E-5	7.5 E-5	4812.1 E-5	196.3 E-5
	0.15	387.6 E-5	7.0 E-5	0.146	700.3 E-5	101.8 E-5	17.1 E-5	4715.7 E-5	348.1 E-5
	Reference	$\|\mathbb{E}[p^g]_{AC}\|_\infty$ 0.5800	$\|\sigma[p^g]_{AC}\|_\infty$ 0.1132	$\|\mathbb{E}[q^g]_{AC}\|_\infty$ 0.3829	$\|\sigma[q^g]_{AC}\|_\infty$ 0.0038	$\|\mathbb{E}[v]_{AC}\|_\infty$ 1.0792	$\|\sigma[v]_{AC}\|_\infty$ 0.0012	$\|\mathbb{E}[i_{i-j}]_{AC}\|_\infty$ 0.3951	$\|\sigma[i_{i-j}]_{AC}\|_\infty$ 0.0100

Errors in Polynomial Chaos and Linearized AC

Linearized AC generally at least one order of magnitude larger errors.

Linearized AC introduces errors in estimating the mean!

Polynomial chaos provides better (but not perfect) approximation of chance constraints.

Table V. EMPIRICAL CONSTRAINT SATISFACTION, COST, AND EXPECTED POWER FLOW VIOLATION FOR MAX. DEGREE $N_d \in \{1, 2\}$ FOR 30-BUS SYSTEM.

		Maximum degree $N_d = 1$							Maximum degree $N_d = 2$						
s	ε	$p_{g,3}^{\max}$	$p_{g,4}^{\max}$	$i_{21\text{-}22}^{\max}$	$i_{15\text{-}23}^{\max}$	$i_{25\text{-}27}^{\max}$	Cost	PF violation $\mathbb{E}[\cdot]/1$ E-3	$p_{g,3}^{\max}$	$p_{g,4}^{\max}$	$i_{21\text{-}22}^{\max}$	$i_{15\text{-}23}^{\max}$	$i_{25\text{-}27}^{\max}$	Cost	PF violation $\mathbb{E}[\cdot]/1$ E-3
0.10	0.05	0.9495	0.9499	0.9436	0.9514	0.9490	599.25	0.0640	0.9494	0.9499	0.9424	0.9526	0.9483	599.25	0.0009
	0.10	0.9026	0.9015	0.8954	0.8959	0.8980	599.24	0.0618	0.9022	0.9012	0.8953	0.8961	0.8980	599.24	0.0007
	0.15	0.8514	0.8507	0.8812	0.8515	0.8489	599.24	0.0601	0.8515	0.8506	0.8808	0.8516	0.8487	599.24	0.0006
0.15	0.05	0.9494	0.9499	0.9388	0.9486	0.9475	599.38	0.1502	0.9494	0.9500	0.9381	0.9511	0.9473	599.38	0.0042
	0.10	0.9028	0.9015	0.8938	0.8937	0.8969	599.36	0.1421	0.9030	0.9015	0.8927	0.8941	0.8969	599.36	0.0029
	0.15	0.8514	0.8502	0.8488	0.8427	0.8486	599.35	0.1373	0.8515	0.8501	0.8484	0.8426	0.8485	599.35	0.0024

Outline

- A complicated model
- A simple chance constraint
- Solution approaches

Interpretation as constraint tightening

$$i(x,0) + f(1-\varepsilon)\sqrt{di_\omega \Sigma_\omega di_\omega} \leq i^{max}$$

$$\underbrace{i(x,0) \leq i^{max}}_{\text{Deterministic constraint}} \underbrace{- f(1-\varepsilon)\sqrt{di_\omega \Sigma_\omega di_\omega}}_{\text{“Uncertainty margin”}}$$

Interpretation as constraint tightening

$$\min_{P_G} \ \sum_{i\in\mathcal{G}}\left(c_{2,i}p_{G,i}^2 + c_{1,i}\,p_{G,i} + c_{0,i}\ \right)$$

s.t.

$$f(\theta, v, p, q) = 0, \quad \forall\, \omega \in U$$

$$p_G \le p_G^{max} - \Phi^{-1}(1-\epsilon)\sqrt{h_P(x)\Sigma_{cov}h_P(x)^T}$$

$$p_G \ge p_G^{min} + \Phi^{-1}(1-\epsilon)\sqrt{h_P(x)\Sigma_{cov}h_P(x)^T}$$

$$i \le i^{max} - \Phi^{-1}(1-\epsilon)\sqrt{h_I(x)\Sigma_{cov}h_I(x)^T}$$

$$v \le v^{max} - \Phi^{-1}(1-\epsilon)\sqrt{h_V(x)\Sigma_{cov}h_V(x)^T}$$

$$\underbrace{v \ge v^{min}}_{\text{Deterministic constraints}} + \underbrace{\Phi^{-1}(1-\epsilon)\sqrt{h_V(x)\Sigma_{cov}h_V(x)^T}}_{\text{"Uncertainty margins"}}$$

An efficient iterative algorithm

- Main idea: Separate optimization and uncertainty assessment

Initialize: $\lambda_i = 0$

Solve deterministic AC OPF (λ_i fixed):	$i(\boldsymbol{x}, 0) \leq i^{max} - \lambda_i$
Evaluate tightening λ_i (x fixed):	$\lambda_i = f(1-\varepsilon)\sqrt{di_\omega \Sigma_\omega di_\omega}$

Constraint

Tightening

Converged to **safe** solution when $\left|\lambda_i^k - \lambda_i^{k-1}\right| \leq \eta$

An efficient iterative algorithm

- Main idea: Separate optimization and uncertainty assessment

Initialize: $\lambda_i = 0$

Solve deterministic AC OPF (λ_i fixed): $i(\boldsymbol{x}, 0) \leq i^{max} - \lambda_i$

Use your favorite AC OPF solver!

Evaluate tightening λ_i (x fixed): $\lambda_i = f(1-\varepsilon)\sqrt{di_\omega \Sigma_\omega di_\omega}$

Tightening

Converged to **safe** solution when $|\lambda_i^k - \lambda_i^{k-1}| \leq \eta$

An efficient iterative algorithm

- Main idea: Separate optimization and uncertainty assessment

Initialize: $\lambda_i = 0$

Solve deterministic AC OPF (λ_i fixed):	$i(\boldsymbol{x}, 0) \leq i^{max} - \lambda_i$
Evaluate tightening λ_i (x fixed):	$\lambda_i =$ Robust, Monte Carlo …

Converged to **safe** solution when $\left|\lambda_i^k - \lambda_i^{k-1}\right| \leq \eta$

Use any method for uncertainty quantification!

Robust bounds on uncertainty impact:

$$\lambda = \max_{\omega \in W} i(x, \omega)$$

[Molzahn and Roald, PSCC ‘18],
[Molzahn and Roald, HICSS ‘18]

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Converged to **safe** solution when $|\lambda_i^k - \lambda_i^{k-1}| \leq \eta$

No guarantees for convergence

[Roald, Molzahn, Tobler '17]

No guarantees for optimality

But works surprisingly well!

[Roald and Andersson '17]

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↓

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No guarantees for convergence

[Roald, Molzahn, Tobler '17]

No guarantees for optimality

But works surprisingly well!

[Roald and Andersson '17]

	RTS96	118 Bus	300 Bus	Polish
Buses	24	118	300	2383
Uncertain loads	17	99	131	941
Solution time	0.54s	1.15s	3.37s	31.89s
Iterations	5	4	5	4

Practical chance constraint implementation

- Implementation tested on the European Grid!

Umbrella Project
www.e-umbrella.eu

Initialize: $\lambda_i = 0$

↓

Solve deterministic AC OPF (λ_i fixed):	$i(\boldsymbol{x}, 0) \leq i^{max} - \lambda_i$

↓

Evaluate tightening λ_i (x fixed):	λ_i = Monte Carlo …

↓

More safe solution than before!